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Maximal superintegrability of the generalized Kepler–Coulomb system on N-dimensional curved spaces

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Abstract
The superposition of the Kepler–Coulomb potential on the 3D Euclidean space with *three* centrifugal terms has recently been shown to be maximally superintegrable (Verrier and Evans 2008 *J. Math. Phys.* **49** 022902) by finding an additional (hidden) integral of motion which is *quartic* in the momenta. In this paper, we present the generalization of this result to the N-dimensional spherical, hyperbolic and Euclidean spaces by making use of a unified symmetry approach that makes use of the curvature parameter. The resulting Hamiltonian, formed by the (curved) Kepler–Coulomb potential together with N centrifugal terms, is shown to be endowed with $2N - 1$ functionally independent integrals of the motion: one of them is quartic and the remaining ones are quadratic. The transition from the proper Kepler–Coulomb potential, with its associated *quadratic* Laplace–Runge–Lenz N-vector, to the generalized system is fully described. The role of spherical, nonlinear (cubic) and coalgebra symmetries in all these systems is highlighted.

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1. Introduction

The Kepler–Coulomb (KC) potential on Riemannian spaces of constant curvature was already studied by Lipschitz and Killing in the nineteenth century, and later rediscovered by Schrödinger [1] (see [2] for a detailed discussion). In terms of a geodesic radial distance r between the particle and the origin of the space, the KC potential on the N-dimensional (ND) spherical $\mathbb{S}^N$, Euclidean $\mathbb{E}^N$ and hyperbolic $\mathbb{H}^N$ spaces reads as (see, e.g., [3–5] and references therein)

$$-\frac{K}{\tan r} \quad \text{on} \quad \mathbb{S}^N, \qquad -\frac{K}{r} \quad \text{on} \quad \mathbb{E}^N, \qquad -\frac{K}{\tanh r} \quad \text{on} \quad \mathbb{H}^N. \tag{1.1}$$

In this paper we shall deal with the integrability properties of the so-called *ND generalized KC system*, which is defined as the superposition of the (curved) KC potential with N 'centrifugal' terms. In the ND Euclidean space $\mathbb{E}^N$, such a system reads as

$$\mathcal{H} = \frac{1}{2}\mathbf{p}^2 - \frac{K}{\sqrt{\mathbf{q}^2}} + \sum_{i=1}^{N} \frac{b_i}{q_i^2}, \tag{1.2}$$

where K and b_i ($i = 1, \ldots, N$) are real constants. This system was known to be *quasi-maximally superintegrable* [3] in the Liouville sense [6], since a set of $2N - 2$ functionally independent *quadratic* integrals of motion (including the Hamiltonian) were explicitly known. In fact, in the remarkable classification on superintegrable systems on $\mathbb{E}^3$ by Evans [7], this Hamiltonian was called 'weakly' or *minimally superintegrable* since it had one integral of motion more (four) than the number required to be completely integrable (three), but one less than the maximum possible number of independent integrals for a 3D system (five).

In contrast, it was also well known that when at least *one* of the centrifugal terms vanishes (we shall call this case the *quasi-generalized KC system*), the resulting Hamiltonian turns out to be *maximally superintegrable* in arbitrary dimension since a maximal set of $2N - 1$ functionally independent and *quadratic* integrals of the motion is explicitly known (see [3, 7–10] and references therein). Moreover, such maximal superintegrability of the quasi-generalized KC system has also been proven for the spherical and hyperbolic spaces [11–13] as well as for the Minkowskian and (anti-)de Sitter spacetimes [4, 14].

Nevertheless, in a recent work Verrier and Evans [15] have shown that the generalized KC system on $\mathbb{E}^3$ (i.e., the superposition of the 3D KC potential with three centrifugal terms) is *maximally superintegrable*, but the additional integral of motion is *quartic* in the momenta. The aim of this paper is to show that this result holds for an arbitrary dimension N and, moreover, that the generalized KC system is also maximally superintegrable on the ND curved Riemannian spaces of constant curvature: the spherical $\mathbb{S}^N$ and hyperbolic $\mathbb{H}^N$ spaces. In this way, the ND Euclidean system arises as a smooth limiting flat case that can be interpreted as a contraction in terms of the curvature parameter κ.

In order to prove this result, we shall explicitly construct the set of $2N - 1$ functionally independent integrals of motion for the generalized KC Hamiltonian. In particular, $2N - 2$ of them will be *quadratic* and provided by an $\mathfrak{sl}(2, \mathbb{R})$ Poisson coalgebra symmetry [3, 16] (together with the Hamiltonian), while the remaining 'hidden' one is *quartic* in the momenta and generalizes the result of [15] on $\mathbb{E}^3$ to these three ND classical spaces of constant curvature. In this way, the short list of ND maximally superintegrable Hamiltonians (see [17] an references therein) is enlarged with another instance.

The paper is organized as follows. In the following section, we introduce the geometric background on which the rest of the paper will be based: the Poincaré and Beltrami phase spaces arising, respectively, as the stereographic and the central projection from a linear ambient space $\mathbb{R}^{N+1}$ [3, 18]. The next section is devoted to recalling the description of the maximal integrability of the curved KC system in terms of these two phase spaces. In section 4 the spherical and 'hidden' nonlinear symmetries of the KC system are fully described, thus providing a detailed explanation of the techniques making possible the 'transition' from the integrability properties of the curved KC system to the generalized one. The core of the paper is contained in section 5, where we explicitly show how the spherical symmetry breaking induced by the centrifugal terms can be appropriately replaced by an $\mathfrak{sl}(2,\mathbb{R})$ Poisson coalgebra symmetry [19, 20] that allows us to construct the 'additional' *quartic* integral of motion for the *N*D generalized curved KC systems. Finally, some remarks close the paper.

2. Poincaré and Beltrami phase spaces

To start with we present the structure of the Poincaré and Beltrami phase spaces [3], which will allow us to deal with the three classical Riemannian spaces in a unified setting. Furthermore, as a new result, we also present the explicit canonical transformation relating both of them.

Given a constant sectional curvature κ, the *N*D spherical $\mathbb{S}^N(\kappa > 0)$, Euclidean $\mathbb{E}^N$ $(\kappa = 0)$ and hyperbolic $\mathbb{H}^N(\kappa < 0)$ spaces can be simultaneously embedded in an ambient linear space $\mathbb{R}^{N+1}$ with ambient (or Weierstrass) coordinates $(x_0,\mathbf{x}) = (x_0, x_1,\dots,x_N)$ by requiring them to fulfil the 'sphere' constraint Σ: $x_0^2 + \kappa\mathbf{x}^2 = 1$. Hereafter for any two *N*D vectors, say $\mathbf{a} = (a_1,\dots,a_N)$ and $\mathbf{b} = (b_1,\dots,b_N)$, we denote $\mathbf{a}^2 = \sum_{i=1}^N a_i^2$, $|\mathbf{a}| = \sqrt{\mathbf{a}^2}$ and $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^N a_i b_i$. The metric on these three Riemannian spaces of constant curvature is given, in ambient coordinates, by [21]

$$\mathrm{d}s^2 = \frac{1}{\kappa}\left(\mathrm{d}x_0^2 + \kappa\,\mathrm{d}\mathbf{x}^2\right)\Big|_{\Sigma}. \tag{2.1}$$

Now, if we consider the stereographic projection [18] from $(x_0,\mathbf{x}) \in \Sigma \subset \mathbb{R}^{N+1}$ to the *Poincaré coordinates* $\mathbf{q}\in\mathbb{R}^N$ with pole $(-1,\mathbf{0})\in\mathbb{R}^{N+1}$, that is, $(-1,\mathbf{0}) + \lambda(1,\mathbf{q}) \in \Sigma$, we obtain that

$$\lambda = \frac{2}{1+\kappa\mathbf{q}^2},\qquad x_0 = \lambda - 1 = \frac{1-\kappa\mathbf{q}^2}{1+\kappa\mathbf{q}^2},\qquad \mathbf{x} = \lambda\mathbf{q} = \frac{2\mathbf{q}}{1+\kappa\mathbf{q}^2}. \tag{2.2}$$

On the other hand, if we apply the central projection from $(x_0,\mathbf{x})$ to the *Beltrami coordinates* $\tilde{\mathbf{q}}\in\mathbb{R}^N$ with pole $(0,\mathbf{0})\in\mathbb{R}^{N+1}$, such that $(0,\mathbf{0}) + \mu(1,\tilde{\mathbf{q}}) \in \Sigma$, we find

$$\mu = \frac{1}{\sqrt{1+\kappa\tilde{\mathbf{q}}^2}},\qquad x_0 = \mu,\qquad \mathbf{x} = \mu\tilde{\mathbf{q}} = \frac{\tilde{\mathbf{q}}}{\sqrt{1+\kappa\tilde{\mathbf{q}}^2}}. \tag{2.3}$$

The image of these projections is the subset of $\mathbb{R}^N$ determined by either $\lambda > 0$ (and then $1+\kappa\mathbf{q}^2 > 0$) or $\mu\in\mathbb{R}$ (and thus $1+\kappa\tilde{\mathbf{q}}^2 > 0$). This means that for $\mathbb{S}^N$ $(\kappa > 0)$, both projections lead to $\mathbb{R}^N$ with the exception of a single point; for $\mathbb{H}^N$, they give the open subset $\mathbf{q}^2 < 1/|\kappa|$ or $\tilde{\mathbf{q}}^2 < 1/|\kappa|$ (the Poincaré disc in 2D); and in both cases, if $\kappa = 0$, we recover $\mathbb{E}^N$ in Cartesian coordinates $\mathbf{x}\in\mathbb{R}^N\equiv\mathbb{E}^N$ since $\mathbf{x} = 2\mathbf{q} = \tilde{\mathbf{q}}$.

Therefore, it can be shown that the metric (2.1) in both coordinate systems reads as

$$\mathrm{d}s^2 = 4\frac{\mathrm{d}\mathbf{q}^2}{(1+\kappa\mathbf{q}^2)^2} = \frac{(1+\kappa\tilde{\mathbf{q}}^2)\,\mathrm{d}\tilde{\mathbf{q}}^2 - \kappa(\tilde{\mathbf{q}}\cdot\mathrm{d}\tilde{\mathbf{q}})^2}{(1+\kappa\tilde{\mathbf{q}}^2)^2} \tag{2.4}$$

so that the corresponding geodesic flow on these spaces can be described through the free Lagrangian given (up to a positive multiplicative constant) by

$$\mathcal{T} = 2\frac{\dot{\mathbf{q}}^2}{(1+\kappa\mathbf{q}^2)^2} = \frac{(1+\kappa\tilde{\mathbf{q}}^2)\dot{\tilde{\mathbf{q}}}^2 - \kappa(\tilde{\mathbf{q}}\cdot\dot{\tilde{\mathbf{q}}})^2}{2(1+\kappa\tilde{\mathbf{q}}^2)^2}. \tag{2.5}$$

Hence, the canonical momenta $\mathbf{p},\tilde{\mathbf{p}}$ conjugate to $\mathbf{q},\tilde{\mathbf{q}}$ are obtained through a Legendre transformation yielding

$$\mathbf{p} = 4\frac{\dot{\mathbf{q}}}{(1+\kappa\mathbf{q}^2)^2},\qquad \tilde{\mathbf{p}} = \frac{(1+\kappa\tilde{\mathbf{q}}^2)\dot{\tilde{\mathbf{q}}} - \kappa(\tilde{\mathbf{q}}\cdot\dot{\tilde{\mathbf{q}}})\tilde{\mathbf{q}}}{(1+\kappa\tilde{\mathbf{q}}^2)^2}, \tag{2.6}$$

and the geodesic flow kinetic energy is found to be

$$\mathcal{T} = \tfrac{1}{8}(1+\kappa\mathbf{q}^2)^2\mathbf{p}^2 = \tfrac{1}{2}(1+\kappa\tilde{\mathbf{q}}^2)(\tilde{\mathbf{p}}^2+\kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})^2). \tag{2.7}$$

Evidently, $\mathcal{T}$ can be written with a factor $1/2$ instead of $1/8$ in the Poincaré phase space but we have kept the latter factor in order to make explicit the equality between Poincaré and Beltrami expressions.

The canonical equivalence between both phase spaces is characterized by the following statement that can be proven through direct computations and by taking into account expressions (2.2)–(2.6).

Proposition 1. *Let* $(\mathbf{q},\mathbf{p})$ *be the Poincaré phase-space variables such that* $\{q_i, p_j\} = \delta_{ij}$ *and* $(\tilde{\mathbf{q}},\tilde{\mathbf{p}})$ *the Beltrami ones satisfying* $\{\tilde{q}_i, \tilde{p}_j\} = \delta_{ij}$*. Both sets of canonical variables are related through the canonical transformation given by*

$$\begin{aligned} \mathbf{q} &= \frac{\tilde{\mathbf{q}}}{1+\sqrt{1+\kappa\tilde{\mathbf{q}}^2}}, \qquad & \mathbf{p} &= \left(1+\sqrt{1+\kappa\tilde{\mathbf{q}}^2}\right)\tilde{\mathbf{p}}+\kappa\tilde{\mathbf{q}}(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}}) \\ \tilde{\mathbf{q}} &= \frac{2\mathbf{q}}{1-\kappa\mathbf{q}^2}, \qquad & \tilde{\mathbf{p}} &= \frac{1-\kappa\mathbf{q}^2}{2(1+\kappa\mathbf{q}^2)}\left((1+\kappa\mathbf{q}^2)\mathbf{p}-2\kappa\mathbf{q}(\mathbf{q}\cdot\mathbf{p})\right). \end{aligned} \tag{2.8}$$

Moreover, from (2.8), we obtain the following useful relations to be considered below:

$$\begin{aligned} & q_ip_j - q_jp_i = \tilde{q}_i\tilde{p}_j - \tilde{q}_j\tilde{p}_i, \qquad \frac{q_i}{q_j} = \frac{\tilde{q}_i}{\tilde{q}_j}, \qquad \frac{q_i}{\sqrt{\mathbf{q}^2}} = \frac{\tilde{q}_i}{\sqrt{\tilde{\mathbf{q}}^2}} \\ & \tilde{\mathbf{p}}+\kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})\tilde{\mathbf{q}} = \frac{1}{2}(1-\kappa\mathbf{q}^2)\mathbf{p}+\kappa(\mathbf{q}\cdot\mathbf{p})\mathbf{q} \\ & \tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}} = \frac{1-\kappa\mathbf{q}^2}{1+\kappa\mathbf{q}^2}(\mathbf{q}\cdot\mathbf{p}). \end{aligned} \tag{2.9}$$

3. The Kepler–Coulomb system

In order to construct the KC Hamiltonian we recall that for the three Riemannian spaces with constant curvature, the geodesic radial distance r (along the geodesic that joins the particle and the origin in the space) can be expressed, in this order, in ambient, Poincaré and Beltrami coordinates as [3]

$$\frac{1}{\kappa}\tan^2(\sqrt{\kappa}r) = \frac{\mathbf{x}^2}{x_0^2} = \frac{4\mathbf{q}^2}{(1-\kappa\mathbf{q}^2)^2} = \tilde{\mathbf{q}}^2. \tag{3.1}$$

In fact, these three expressions provide an appropriate definition for the curved (Higgs) oscillator potential [22, 23].

By using these coordinates, the KC system is just the Hamiltonian [3]:

$$\begin{aligned} \mathcal{H} &= \frac{1}{8}(1+\kappa\mathbf{q}^2)^2\mathbf{p}^2 - K\frac{1-\kappa\mathbf{q}^2}{2\sqrt{\mathbf{q}^2}} \\ &= \frac{1}{2}(1+\kappa\tilde{\mathbf{q}}^2)(\tilde{\mathbf{p}}^2+\kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})^2) - \frac{K}{\sqrt{\tilde{\mathbf{q}}^2}}, \end{aligned} \tag{3.2}$$

where the first term is the kinetic energy (2.7) and the second one is the curved KC potential, which is obtained as the square root of the inverse of (3.1). We stress that under this framework, we are able to cover simultaneously the three cases (1.1) for the particular cases of the sectional curvature $\kappa \in \{+1, 0, -1\}$. Moreover, in this language the limit $\kappa \to 0$ corresponds to the (flat) contraction $\mathbb{S}^N \to \mathbb{E}^N \leftarrow \mathbb{H}^N$.

The maximal superintegrability of the curved KC Hamiltonian (3.2) is characterized by the following proposition.

Proposition 2. *[3] Let $\mathcal{H}$ be the KC Hamiltonian (3.2) and let us consider the quadratic functions in the momenta given by*

$$C^{(m)} = \sum_{1\leqslant i<j}^{m} (q_i p_j - q_j p_i)^2, \qquad C_{(m)} = \sum_{N-m+1\leqslant i<j}^{N} (q_i p_j - q_j p_i)^2 \tag{3.3}$$

$$\begin{aligned}\mathcal{L}_i &= \sum_{l=1}^{N} \left(\frac{1}{2}(1-\kappa\mathbf{q}^2)p_l + \kappa(\mathbf{q}\cdot\mathbf{p})q_l\right)(q_l p_i - q_i p_l) + K\frac{q_i}{\sqrt{\mathbf{q}^2}} \\ &= \sum_{l=1}^{N} (\tilde{p}_l + \kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})\tilde{q}_l)(\tilde{q}_l\tilde{p}_i - \tilde{q}_i\tilde{p}_l) + K\frac{\tilde{q}_i}{\sqrt{\tilde{\mathbf{q}}^2}},\end{aligned} \tag{3.4}$$

where $m = 2,\ldots,N$, $C^{(N)} = C_{(N)}$ and $i = 1,\ldots,N$. Then,

(i) the $2N-3$ functions (3.3) and the N functions (3.4) Poisson-commute with $\mathcal{H}$,

(ii) each set $\{\mathcal{H}, C^{(m)}\}$ and $\{\mathcal{H}, C_{(m)}\} (m = 2,\ldots,N)$ provides N functionally independent functions in involution,

(iii) for a fixed i, the $2N-1$ functions $\{\mathcal{H}, C^{(m)}, C_{(m)}, \mathcal{L}_i\}$ with $m = 2,\ldots,N$ are functionally independent.

In order to illustrate this result, we point out that the Euclidean KC system is obtained from proposition 2 by setting $\kappa = 0$. In this case, the set of integrals $C^{(m)}$ and $C_{(m)}$ does not change, as it reflects the 'abstract' spherical symmetry of the system. In contrast, the $\mathcal{L}_i$ integral reads as

$$\mathcal{L}_i = \sum_{l=1}^{N}\frac{1}{2}p_l(q_l p_i - q_i p_l) + K\frac{q_i}{\sqrt{\mathbf{q}^2}} = \sum_{l=1}^{N}\tilde{p}_l(\tilde{q}_l\tilde{p}_i - \tilde{q}_i\tilde{p}_l) + K\frac{\tilde{q}_i}{\sqrt{\tilde{\mathbf{q}}^2}}. \tag{3.5}$$

Of course, this is indeed the well-known expression for the ith component of the Laplace–Runge–Lenz vector on $\mathbb{E}^N$; note that the only difference between both expressions is the $1/2$ factor for the Poincaré variables since $\mathbf{q} = \frac{1}{2}\tilde{\mathbf{q}}$ and $\mathbf{p} = 2\tilde{\mathbf{p}}$ when $\kappa = 0$ as it follows from (2.8).

4. Symmetries of the Kepler–Coulomb system

At this point, a detailed analysis of the symmetry properties of the integrals of the motion for the KC Hamiltonian is worth performing, since this background will provide the appropriate framework to extend such integrability properties to the generalized KC system.

4.1. Spherical symmetry

First, we stress that the constants of motion (3.3) keep the same form in both Poincaré and Beltrami phase spaces (see (2.9)). They reflect the well-known spherical symmetry of the KC system (and of any central potential as well). In particular, the functions $J_{ij} = q_i p_j - q_j p_i$ with $i < j$ and $i, j = 1,\ldots,N$ span an $\mathfrak{so}(N)$ Lie–Poisson algebra

$$\{J_{ij}, J_{ik}\} = J_{jk}, \qquad \{J_{ij}, J_{jk}\} = -J_{ik}, \qquad \{J_{ik}, J_{jk}\} = J_{ij}, \qquad i<j<k. \tag{4.1}$$

Thus, the constants of motion (3.3) are the Casimirs of certain rotation subalgebras $\mathfrak{so}(m) \subset \mathfrak{so}(N)$ written through the sums of the square of *some* angular momentum components J_{ij}. The square of the total angular momentum $\mathbf{J}^2$ is then given by the Casimir of $\mathfrak{so}(N)$:

$$\mathbf{J}^2 = C^{(N)} = C_{(N)} = \sum_{1\leqslant i<j}^{N} J_{ij}^2. \tag{4.2}$$

This $\mathfrak{so}(N)$-symmetry can be further enlarged by defining the following N functions P_i $(i = 1, \ldots, N)$:

$$P_i = \tfrac{1}{2}(1 - \kappa\mathbf{q}^2)p_i + \kappa(\mathbf{q}\cdot\mathbf{p})q_i = \tilde{p}_i + \kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})\tilde{q}_i, \tag{4.3}$$

which come from the first factors of the integrals $\mathcal{L}_i$ (3.4) (so these are specifically characterized by the proper KC system). Their Lie–Poisson brackets are given by

$$\{J_{ij}, P_k\} = \delta_{ik}P_j - \delta_{jk}P_i, \qquad \{P_i, P_j\} = \kappa J_{ij}. \tag{4.4}$$

Therefore, the $N(N+1)/2$ functions $\langle J_{ij}, P_i\rangle (i < j; i, j = 1, \ldots, N)$ span a Lie–Poisson algebra $\mathfrak{so}_\kappa(N+1)$ [14, 21], with commutation relations given by (4.1) and (4.4), in which κ can be interpreted as a contraction parameter. These Poisson brackets define the spherical $\mathfrak{so}(N+1)$ for $\kappa > 0$, hyperbolic $\mathfrak{so}(N, 1)$ for $\kappa < 0$ and the Euclidean Lie–Poisson algebra $\mathfrak{iso}(N)$ for $\kappa = 0$.

Moreover, the three Riemannian spaces of constant sectional curvature κ, described in section 2, can be constructed as homogeneous spaces through the quotient $\langle J_{ij}, P_i\rangle/\langle J_{ij}\rangle = \langle P_i\rangle$:

$$\begin{aligned} \mathbb{S}^N &= \mathfrak{so}(N+1)/\mathfrak{so}(N) && \text{for} \quad \kappa > 0,\\ \mathbb{E}^N &= \mathfrak{iso}(N)/\mathfrak{so}(N) && \text{for} \quad \kappa = 0,\\ \mathbb{H}^N &= \mathfrak{so}(N,1)/\mathfrak{so}(N) && \text{for} \quad \kappa < 0, \end{aligned} \tag{4.5}$$

with P_i's playing the role of (curved) translations on such spaces, and the contraction $\kappa = 0$ gives rise to the Euclidean (commutative) translations $P_i = \frac{1}{2}p_i = \tilde{p}_i$. The Poisson brackets between the Hamiltonian $\mathcal{H}$ (3.2) and the $\mathfrak{so}_\kappa(N+1)$ generators read as

$$\{\mathcal{H}, J_{ij}\} = 0, \qquad \{\mathcal{H}, P_i\} = K\frac{q_i(1+\kappa\mathbf{q}^2)^2}{4|\mathbf{q}|^3} = K\frac{\tilde{q}_i(1+\kappa\tilde{\mathbf{q}}^2)}{|\tilde{\mathbf{q}}|^3} \equiv K\frac{x_i}{\mathbf{x}^3}. \tag{4.6}$$

We stress that from the viewpoint of such $\mathfrak{so}_\kappa(N+1)$-symmetry, the constants of motion (3.4) can be expressed in a very natural and simple form, as [14]

$$\mathcal{L}_i = \sum_{l=1}^{N} P_l J_{li} + K\frac{q_i}{\sqrt{\mathbf{q}^2}} \tag{4.7}$$

such that $J_{ii} \equiv 0$ and $J_{li} = -J_{il}$ if $l > i$. This, in turn, directly shows the functional independence of a given $\mathcal{L}_i$ with respect to the integrals (3.3), as stated in proposition 2, since the latter are only formed by rotation generators J_{ij} of $\mathfrak{so}(N)$.

4.2. Nonlinear angular momentum symmetry

By taking into account (4.7), it is a matter of straightforward computations to show that the N constants of motion (3.4) are transformed as an N-vector under the $\mathfrak{so}(N)$ generators (4.1):

$$\{J_{ij}, \mathcal{L}_k\} = \delta_{ik}\mathcal{L}_j - \delta_{jk}\mathcal{L}_i. \tag{4.8}$$

The N functions $\mathcal{L}_i$ are, in fact, the components of the *Laplace–Runge–Lenz N-vector* on $\mathbb{S}^N$, $\mathbb{H}^N$ and $\mathbb{E}^N$, which correspond to the 'hidden' symmetries of the KC Hamiltonian. Moreover, the Lie–Poisson brackets involving the $\mathcal{L}_i$ components read as

$$\{\mathcal{L}_i, \mathcal{L}_j\} = 2(\kappa\mathbf{J}^2 - \mathcal{H})J_{ij} \equiv \Lambda J_{ij}. \tag{4.9}$$

This expression is worth comparing with (4.4). Since $\mathcal{H}$ Poisson-commutes with all the functions J_{ij} and $\mathcal{L}_i$ (in algebraic terms, we would say that $\mathcal{H}$ behaves as a central extension), we find that the set of $N(N+1)/2$ functions $\langle J_{ij}, \mathcal{L}_i\rangle (i < j; i, j = 1, \ldots, N)$ span a *nonlinear* (cubic) Poisson algebra that we will denote as $\mathfrak{so}_\kappa^{(3)}(N+1)$. Only in $\mathbb{E}^N$ ($\kappa = 0$) does this nonlinear symmetry algebra reduce to Lie–Poisson algebras with $\Lambda = -2\mathcal{H}$ being a constant, and we get $\mathfrak{so}(N+1)$ for $\mathcal{H} < 0$ or $\mathfrak{so}(N, 1)$ for $\mathcal{H} > 0$.

Therefore, from a geometrical viewpoint we can say that the role of the translations P_i on the homogeneous spaces (4.5), with Lie–Poisson symmetry $\mathfrak{so}_\kappa(N+1)$, is replaced by the Laplace–Runge–Lenz components $\mathcal{L}_i$, with nonlinear symmetry $\mathfrak{so}_\kappa^{(3)}(N+1)$, whereas the role of the former constant curvature κ is now played by the *quadratic function* Λ (4.9). In this respect, note that although $\mathbf{J}^2$ and $\mathcal{H}$ are both constants of the motion for the KC system, $\mathbf{J}^2$ does not Poisson-commute with $\mathcal{L}_i$. As a consequence, Λ is not a central function within $\mathfrak{so}_\kappa^{(3)}(N+1)$.

However, in the case $N = 2$ we have that $\mathbf{J}^2 \equiv J_{12}^2$, so that $\mathfrak{so}_\kappa^{(3)}(3) = \langle J_{12}, \mathcal{L}_1, \mathcal{L}_2\rangle$ gives rise to the *cubic* Poisson algebra

$$\{J_{12}, \mathcal{L}_1\} = \mathcal{L}_2, \qquad \{J_{12}, \mathcal{L}_2\} = -\mathcal{L}_1, \qquad \{\mathcal{L}_1, \mathcal{L}_2\} = 2\kappa J_{12}^3 - 2\mathcal{H} J_{12}. \tag{4.10}$$

Next, we can define 'number' $\mathcal{L}$ and 'ladder' operators $\mathcal{L}_\pm$ as

$$\mathcal{L} = \mathrm{i} J_{12}, \qquad \mathcal{L}_\pm = \mathcal{L}_1 \pm \mathrm{i}\mathcal{L}_2,$$

fulfilling the cubic commutation relations

$$\{\mathcal{L}, \mathcal{L}_\pm\} = \pm\mathcal{L}_\pm, \qquad \{\mathcal{L}_+, \mathcal{L}_-\} = 4\kappa\mathcal{L}^3 + 4\mathcal{H}\mathcal{L}, \tag{4.11}$$

which reproduce the Poisson algebra analogue of the Higgs $\mathfrak{sl}^{(3)}(2, \mathbb{R})$ algebra [22] whenever $\kappa \neq 0$. In this case the Poisson brackets are associated with $\mathbb{S}^2$ and $\mathbb{H}^2$, whereas the contraction $\kappa = 0$ gives $\mathfrak{sl}(2, \mathbb{R})$ (or $\mathfrak{gl}(2)$ if $\mathcal{H}$ is considered as an actual generator) for $\mathbb{E}^2$. Note that the Higgs algebra is endowed with a *quartic* Casimir function given by

$$\mathcal{C}_{\mathfrak{sl}^{(3)}(2,\mathbb{R})} = \mathcal{L}_+\mathcal{L}_- + \kappa\mathcal{L}^4 + 2\mathcal{H}\mathcal{L}^2. \tag{4.12}$$

Moreover, if we realize this Casimir in terms of Poincaré or Beltrami coordinates, we obtain that $\mathcal{C}_{\mathfrak{sl}^{(3)}(2,\mathbb{R})} = K^2$, which is just the square of the coupling constant of the KC potential. It is worth to stress that the Higgs algebra has been deeply studied and applied to different quantum physical models (beyond integrable systems) with an underlying nonlinear angular momentum symmetry (see [24–30] and references therein).

5. The generalized Kepler–Coulomb system

The generalized KC Hamiltonian $\mathcal{H}^{\mathrm{g}}$ is obtained by adding N centrifugal terms (with non-vanishing parameters $b_i \in \mathbb{R}$) to the KC system $\mathcal{H}$ (3.2). In the curved cases, the way to define appropriately such centrifugal terms was presented and fully explained in [3]. Explicitly, in Poincaré coordinates the generalized KC system is given by

$$\begin{aligned} \mathcal{H}^{\mathrm{g}} &= \mathcal{H} + \frac{1}{8}(1+\kappa\mathbf{q}^2)^2 \sum_{i=1}^{N} \frac{b_i}{q_i^2} \\ &= \frac{1}{8}(1+\kappa\mathbf{q}^2)^2\mathbf{p}^2 - K\frac{1-\kappa\mathbf{q}^2}{2\sqrt{\mathbf{q}^2}} + \frac{1}{8}(1+\kappa\mathbf{q}^2)^2 \sum_{i=1}^{N} \frac{b_i}{q_i^2} \end{aligned} \tag{5.1}$$

while in terms of Beltrami variables, this reads as

$$\mathcal{H}^{\mathrm{g}} = \mathcal{H} + \frac{1}{2}(1+\kappa\tilde{\mathbf{q}}^2)\sum_{i=1}^{N}\frac{b_i}{\tilde{q}_i^2}$$
$$= \frac{1}{2}(1+\kappa\tilde{\mathbf{q}}^2)(\tilde{\mathbf{p}}^2+\kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})^2) - \frac{K}{\sqrt{\tilde{\mathbf{q}}^2}} + \frac{1}{2}(1+\kappa\tilde{\mathbf{q}}^2)\sum_{i=1}^{N}\frac{b_i}{\tilde{q}_i^2}. \qquad (5.2)$$

This is the curved κ-analogue of the generalized KC system on $\mathbb{E}^N$ (1.2). We recall that the centrifugal terms are proper centrifugal barriers on both $\mathbb{E}^N$ ($\kappa = 0$) and $\mathbb{H}^N$ ($\kappa < 0$). Moreover, only on $\mathbb{S}^N$ ($\kappa > 0$) can all these terms be alternatively interpreted as non-central harmonic oscillators (see [4, 21, 31, 32] for a full discussion on the subject).

The maximal superintegrability of this Hamiltonian, which constitutes the main result of this paper, can now be stated and proven as follows.

Theorem 1. *Let $\mathcal{H}^{\mathrm{g}}$ be the generalized KC Hamiltonian (5.1) and (5.2) with all $b_i \neq 0$. Let us consider the quadratic and quartic functions in the momenta given by*

$$C_{\mathrm{g}}^{(m)} = \sum_{1\leqslant i<j}^{m}\left\{(q_ip_j-q_jp_i)^2+\left(b_i\frac{q_j^2}{q_i^2}+b_j\frac{q_i^2}{q_j^2}\right)\right\}+\sum_{i=1}^{m}b_i$$
$$C^{\mathrm{g}}_{(m)} = \sum_{N-m+1\leqslant i<j}^{N}\left\{(q_ip_j-q_jp_i)^2+\left(b_i\frac{q_j^2}{q_i^2}+b_j\frac{q_i^2}{q_j^2}\right)\right\}+\sum_{i=N-m+1}^{N}b_i \qquad (5.3)$$

$$\mathcal{L}_i^{\mathrm{g}} = \left(\sum_{l=1}^{N}\left(\frac{1}{2}(1-\kappa\mathbf{q}^2)p_l+\kappa(\mathbf{q}\cdot\mathbf{p})q_l\right)(q_lp_i-q_ip_l)+\frac{Kq_i}{\sqrt{\mathbf{q}^2}}-(1-\kappa\mathbf{q}^2)\sum_{l=1}^{N}\frac{b_lq_i}{2q_l^2}\right)^2$$
$$+\frac{b_i}{q_i^2}\left(\sum_{l=1}^{N}\left(\frac{1}{2}(1-\kappa\mathbf{q}^2)p_l+\kappa(\mathbf{q}\cdot\mathbf{p})q_l\right)q_l\right)^2$$
$$=\left(\sum_{l=1}^{N}(\tilde{p}_l+\kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})\tilde{q}_l)(\tilde{q}_l\tilde{p}_i-\tilde{q}_i\tilde{p}_l)+\frac{K\tilde{q}_i}{\sqrt{\tilde{\mathbf{q}}^2}}-\sum_{l=1}^{N}\frac{b_l}{\tilde{q}_l^2}\tilde{q}_i\right)^2$$
$$+\frac{b_i}{\tilde{q}_i^2}\left(\sum_{l=1}^{N}(\tilde{p}_l+\kappa(\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}})\tilde{q}_l)\tilde{q}_l\right)^2, \qquad (5.4)$$

where $m = 2,\ldots,N$, $C_{\mathrm{g}}^{(N)} = C^{\mathrm{g}}_{(N)}$ and $i = 1,\ldots,N$. Then,

(i) the $2N-3$ functions (5.3) and the N functions (5.4) Poisson-commute with $\mathcal{H}^{\mathrm{g}}$,
(ii) each set $\{\mathcal{H}^{\mathrm{g}}, C_{\mathrm{g}}^{(m)}\}$ and $\{\mathcal{H}^{\mathrm{g}}, C^{\mathrm{g}}_{(m)}\}$ $(m = 2,\ldots,N)$ provides N functionally independent functions in involution,
(iii) for a fixed i, the $2N-1$ functions $\{\mathcal{H}^{\mathrm{g}}, C_{\mathrm{g}}^{(m)}, C^{\mathrm{g}}_{(m)}, \mathcal{L}_i^{\mathrm{g}}\}$ with $m = 2,\ldots,N$ are functionally independent.

Proof. We shall proceed in two steps. First, we shall prove that (1) $\mathcal{H}^{\mathrm{g}}$ Poisson-commutes with all the integrals $C_{\mathrm{g}}^{(m)}$ and $C^{\mathrm{g}}_{(m)}$ (5.3) and (2) the sets $\{\mathcal{H}^{\mathrm{g}}, C_{\mathrm{g}}^{(m)}\}$ and $\{\mathcal{H}^{\mathrm{g}}, C^{\mathrm{g}}_{(m)}\}$ $(m = 2,\ldots,N)$ are formed by N functionally independent functions in involution.

These two statements can be immediately proven by making use of the $\mathfrak{sl}(2,\mathbb{R})$-coalgebra symmetry [3, 16, 19, 20] of the generalized KC Hamiltonian. Indeed, this coalgebra symmetry is just the appropriate generalization of the spherical symmetry for this problem. In particular, let us define the functions

$$J_- = \mathbf{q}^2, \qquad J_3 = \mathbf{q}\cdot\mathbf{p}, \qquad J_+ = \mathbf{p}^2 + \sum_{i=1}^{N} \frac{b_i}{q_i^2}, \tag{5.5}$$

where $\mathbf{q}$ and $\mathbf{p}$ are, in principle, 'abstract' canonical variables (in this paper, they can be directly identified with either Poincaré or Beltrami ones). These three functions span the $\mathfrak{sl}(2,\mathbb{R})$ Poisson coalgebra with Poisson brackets, coproduct and Casimir function given by

$$\begin{aligned} &\{J_3, J_+\} = 2J_+, \qquad \{J_3, J_-\} = -2J_-, \qquad \{J_-, J_+\} = 4J_3, \\ &\Delta(J_l) = J_l \otimes 1 + 1 \otimes J_l, \qquad l = +, -, 3 \\ &\mathcal{C} = J_- J_+ - J_3^2. \end{aligned} \tag{5.6}$$

We say that the generalized KC Hamiltonian is endowed with this symmetry because it can be written in terms of the $\mathfrak{sl}(2,\mathbb{R})$-coalgebra generators, either in Poincaré variables (5.1), as

$$\mathcal{H}^{\rm g} = \frac{1}{8}\,(1+\kappa J_-)^2\, J_+ - K\,\frac{1-\kappa J_-}{2\sqrt{J_-}} \tag{5.7}$$

or, in Beltrami ones (5.2), as

$$\mathcal{H}^{\rm g} = \frac{1}{2}(1+\kappa J_-)\big(J_+ + \kappa J_3^2\big) - \frac{K}{\sqrt{J_-}}. \tag{5.8}$$

This, in turn, shows that the appearance of centrifugal terms is deeply related to the $\mathfrak{sl}(2,\mathbb{R})$ symplectic realization (5.5) through generator J_+. Once this symmetry for $\mathcal{H}^{\rm g}$ has been proven, the statements (1) and (2) are a direct consequence of such coalgebra invariance (see [3, 16, 20] for a detailed explanation). In particular, $\mathcal{H}^{\rm g}$ (5.7) Poisson-commutes with all the integrals $C_{\rm g}^{(m)}$ and $C^{\rm g}_{(m)}$ (5.3) since the latter are just the left and right mth coproducts of the Casimir (5.6), respectively, under the symplectic realization (5.5). On the other hand, the sets $\big\{\mathcal{H}^{\rm g}, C_{\rm g}^{(m)}\big\}$ and $\big\{\mathcal{H}^{\rm g}, C^{\rm g}_{(m)}\big\}$ $(m = 2,\dots,N)$ are, by construction, given by N functionally independent integrals of the motion in involution.

Now the second task to complete the proof concerns the 'additional' quartic integral $\mathcal{L}_i^{\rm g}$ (5.4). The proof that this Poisson-commutes with $\mathcal{H}^{\rm g}$ can be achieved through direct but cumbersome computations, which can be carried out by starting with expression (5.1) (or (5.2)) and by using the results given in the previous section. Finally, the functional independence of $\mathcal{L}_i^{\rm g}$ with respect to the set $\big\{\mathcal{H}^{\rm g}, C_{\rm g}^{(m)}, C^{\rm g}_{(m)}\big\}$ is automatically fulfilled by considering all these quantities as N-parametric smooth deformations in $\mathbf{b} = (b_1,\dots,b_N)$ of the corresponding non-generalized objects, i.e.

$$\mathcal{H}^{\rm g} = \mathcal{H} + o(\mathbf{b}), \qquad C_{\rm g}^{(m)} = C^{(m)} + o(\mathbf{b}), \qquad C^{\rm g}_{(m)} = C_{(m)} + o(\mathbf{b}), \qquad \mathcal{L}_i^{\rm g} = \mathcal{L}_i + o(\mathbf{b}). \tag{5.9}$$

Hence the proof of the theorem follows by taking into account proposition 2, since we know that $\mathcal{L}_i$ is functionally independent with respect to the set $\{\mathcal{H}, C^{(m)}, C_{(m)}\}$. □

Some remarks are in order as follows.

- The constants of motion (5.3) again keep the same form both in Poincaré and Beltrami variables due to relations (2.9).
- The 'hidden' constants of the motion $\mathcal{L}_i^{\rm g}$ (which are quartic functions in the momenta) can be written in terms of the rotation generators J_{ij} (4.1), the translation ones P_i (4.3)

and the components of the Laplace–Runge–Lenz N-vector $\mathcal{L}_i$ (3.4) associated with the KC system, namely

$$\mathcal{L}_i^{\rm g} = \left(\sum_{l=1}^N P_l J_{li} + \frac{K q_i}{\sqrt{\mathbf{q}^2}} - (1-\kappa \mathbf{q}^2) \sum_{l=1}^N \frac{b_l q_i}{2 q_l^2} \right)^2 + \frac{b_i}{q_i^2} \left(\sum_{l=1}^N P_l q_l \right)^2$$
$$= \left(\mathcal{L}_i - (1-\kappa \mathbf{q}^2) \sum_{l=1}^N \frac{b_l q_i}{2 q_l^2} \right)^2 + \frac{b_i}{q_i^2} \left(\sum_{l=1}^N P_l q_l \right)^2 . \tag{5.10}$$

- The spherical symmetry breaking due to centrifugal terms implies that $\{\mathcal{H}^{\rm g}, J_{ij}\} \neq 0$ and that $\mathbf{J}^2$ (4.2) is no longer a constant of motion (its role is now played by $C_{\rm g}^{(N)} = C_{(N)}^{\rm g}$). In fact, the Poisson brackets (4.6) are now generalized as

$$\{\mathcal{H}^{\rm g}, J_{ij}\} = \frac{b_i q_j^4 - b_j q_i^4}{4 q_i^3 q_j^3} (1+\kappa \mathbf{q}^2)^2 = \frac{b_i \tilde q_j^4 - b_j \tilde q_i^4}{\tilde q_i^3 \tilde q_j^3} (1+\kappa \tilde{\mathbf{q}}^2)$$
$$\{\mathcal{H}^{\rm g}, P_i\} = \frac{2 K q_i^4 - b_i |\mathbf{q}|^3 (1-\kappa \mathbf{q}^2)}{8 q_i^3 |\mathbf{q}|^3} (1+\kappa \mathbf{q}^2)^2 = \frac{K \tilde q_i^4 - b_i |\tilde{\mathbf{q}}|^3}{\tilde q_i^3 |\tilde{\mathbf{q}}|^3} (1+\kappa \tilde{\mathbf{q}}^2). \tag{5.11}$$

As a byproduct of the above results, we straightforwardly recover the quasi-generalized KC systems [3].

Corollary. *Let $\mathcal{H}_i^{\rm g}$ be the quasi-generalized KC Hamiltonian (5.1) and (5.2) with a single $b_i = 0$ (so the index i is fixed) and let us consider the quadratic functions in the momenta (5.3) and*

$$\mathcal{L}_i^{\rm qg} = \sum_{l=1}^N \left(\frac{1}{2} (1-\kappa \mathbf{q}^2) p_l + \kappa (\mathbf{q} \cdot \mathbf{p}) q_l \right) (q_l p_i - q_i p_l)$$
$$+ K \frac{q_i}{\sqrt{\mathbf{q}^2}} - (1-\kappa \mathbf{q}^2) \sum_{l=1; l\neq i}^N \frac{b_l}{2 q_l^2} q_i$$
$$= \sum_{l=1}^N (\tilde p_l + \kappa (\tilde{\mathbf{q}} \cdot \tilde{\mathbf{p}}) \tilde q_l)(\tilde q_l \tilde p_i - \tilde q_i \tilde p_l) + K \frac{\tilde q_i}{\sqrt{\tilde{\mathbf{q}}^2}} - \sum_{l=1; l\neq i}^N \frac{b_l}{\tilde q_l^2} \tilde q_i . \tag{5.12}$$

Then,

(i) $\mathcal{H}_i^{\rm g}$ Poisson-commutes with the $2N-3$ functions (5.3) and the N functions (5.12),

(ii) each set $\{\mathcal{H}_i^{\rm g}, C_{\rm g}^{(m)}\}$ and $\{\mathcal{H}_i^{\rm g}, C_{(m)}^{\rm g}\}(m = 2, \ldots, N)$ is formed by N functionally independent functions in involution,

(iii) the $2N-1$ functions $\{\mathcal{H}_i^{\rm g}, C_{\rm g}^{(m)}, C_{(m)}^{\rm g}, \mathcal{L}_i^{\rm qg}\}$ with $m = 2, \ldots, N$ are functionally independent.

Obviously, if a second parameter b_j also vanishes we obtain an additional integral of motion $\mathcal{L}_j^{\rm qg}$ for the Hamiltonian (that we can now call $\mathcal{H}_{ij}^{\rm g}$). In that case, we have

$$\{\mathcal{H}_{ij}^{\rm g}, J_{ij}\} = \{\mathcal{H}_{ij}^{\rm g}, \mathcal{L}_i^{\rm qg}\} = \{\mathcal{H}_{ij}^{\rm g}, \mathcal{L}_j^{\rm qg}\} = 0, \tag{5.13}$$

and the rotation generator J_{ij} becomes a constant of the motion (see (5.11)). Moreover, the three functions $\langle J_{ij}, \mathcal{L}_i^{\rm qg}, \mathcal{L}_j^{\rm qg}\rangle$ fulfil the Poisson brackets

$$\{J_{ij}, \mathcal{L}_i^{\rm qg}\} = \mathcal{L}_j^{\rm qg}, \qquad \{J_{ij}, \mathcal{L}_i^{\rm qg}\} = -\mathcal{L}_j^{\rm qg},$$
$$\{\mathcal{L}_i^{\rm qg}, \mathcal{L}_j^{\rm qg}\} = 2(\kappa C_{\rm g}^{(N)} - \mathcal{H}_{ij}^{\rm g}) J_{ij} \tag{5.14}$$

to be compared with (4.8) and (4.9). Note that these brackets do not close a nonlinear Poisson algebra due to $C_{\rm g}^{(N)}$ (5.3). The very same process follows when yet one more centrifugal parameter vanishes and so on. From this viewpoint, the KC system (3.2) arises as the 'degenerate' case with all $b_i = 0$ of the generalized one (5.1) and (5.2) in the sense that $\mathcal{H}^{\rm g} \to \mathcal{H}^{\rm g}_{12\ldots N} \equiv \mathcal{H}$ and $\mathcal{L}_i^{\rm g} \to \mathcal{L}_i^{\rm qg} \to \mathcal{L}_i (i = 1, \ldots, N)$.

Finally, we would like to point out that the results presented here on the generalized KC system on curved spaces make this system 'closer' to the Smorodinsky–Winternitz system (i.e., the superposition of the curved harmonic oscillator potential (3.1) with N centrifugal terms) on such spaces [4, 7, 9, 11–14, 21, 33–37]. Both of them are maximally superintegrable, and the only structural difference between them is the fact that all the integrals of the Smorodinsky–Winternitz system are quadratic in the momenta.

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